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A note on nonperturbative renormalization of effective field theory

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Abstract

Within the realm of contact potentials, the key structures intrinsic of nonperturbative renormalization of T-matrices are unraveled using rigorous solutions and an inverse form of the algebraic Lippmann–Schwinger equation. The intrinsic mismatches between effective field theory power counting and nonperturbative divergence structures are shown for the first time to preclude the conventional counterterm algorithm from working in the renormalization of EFT for NN scattering in nonperturbative regimes.

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1. Introduction

The effective field theory (EFT) approach to nucleon systems has been producing many encouraging results [1], pointing toward a promising field-theoretical framework for nuclear forces. In the course of this evidence has been accumulated that the conventional renormalization programs cease to apply in a straightforward manner for such nonperturbative problems, along with debates concerning this issue [2, 3]. This is not totally unexpected as the issue is nonperturbative which may pervert the wisdom established within perturbative frameworks. For example, as noted in [4], perturbative analysis of counterterms [5] can be misleading; therefore, 'new theoretical ideas' for nonperturbative treatment of EFT are needed. The nonperturbative aspects of this issue are also emphasized in [6].

Actually, the difficulties encountered so far even brought about some doubts concerning the validity of the EFT approach to the nuclear systems. In our view, it is natural to think of a field-theoretical approach to nuclear systems as an important advancement, while the difficulties imply that such a treatment has not been fully accomplished yet. Therefore, it is important to unravel hidden structures and notions underlying the nonperturbative renormalization of EFT for nucleon–nucleon (NN) interactions. For this purpose, we will work with contact potentials or pionless EFT to obtain rigorous solutions that could make things transparent. Here, we recall that our main purpose here is not to reproduce the well-known results about pionless EFT in the literature, say [5, 7], but to explore the nonperturbative properties or structures that should be generally useful for nonperturbative renormalization of both pionless and pionful EFTs, and even other effective theories.

2. Parametrization and rigorous solutions

The setup is as follows: the potentials for NN scattering are first systematically constructed using chiral perturbation theory (χPT) up to some chiral order Δ and then resummed through Lippmann–Schwinger equations (LSEs) to obtain the T-matrices [8]. In the case of contact potentials, the LSEs could be turned into algebraic ones using the following trick or ansatz [9, 10] (we consider an uncoupled partial wave channel L for simplicity):

$$V_L(q,q') = q^L q'^L \sum_{i,j=0,1,2,\ldots} \lambda_{ij} q^{2i} q'^{2j} = q^L q'^L U^T(q^2)\lambda U(q'^2), \tag{1}$$

$$T_L(q,q') = q^L q'^L \sum_{i,j=0,1,2,\ldots} \tau_{ij} q^{2i} q'^{2j} = q^L q'^L U^T(q^2)\tau U(q'^2), \tag{2}$$

with q, q' being the external momenta and $U^T(q^2) \equiv (1, q^2, q^4, \ldots)$. Here λ is energy independent while τ is energy dependent[1]. As V_L is truncated at a finite order Δ according to the EFT power counting, we have the following constraints:

$$\lambda_{ij} = 0, \qquad \forall i, j : i + j > \Delta/2 - L. \tag{3}$$

This constraint will prove to be crucial. The algebraic LSE for the channel L now reads

$$\tau(E) = \lambda + \lambda \circ \mathcal{I}(E) \circ \tau(E), \tag{4}$$

with

$$\mathcal{I}(E) \equiv (\mathcal{I}_{ij}(E)), \mathcal{I}_{ij}(E) \equiv \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{k^{2(i+j)}}{E - k^2/M + \mathrm{i}\epsilon}, \qquad i, j = 0, 1, 2, \ldots. \tag{5}$$

So, the renormalization of Ts boils down to the renormalization of τs as $U(q^2)$ or $U^T(q^2)$ is not subject to renormalization at all. Our analysis here is also illuminating for the more realistic cases with pion exchanges, as the LSE there is still dominated by power-like divergences: $V(q, q', \ldots) \sim \sum q^\alpha q'^\beta$ when $q, q' \to \infty$. We note in passing that the above theoretical setup may also be applied to other problems dominated by singular short-range interactions.

Now we parametrize the divergent integrals $[\mathcal{I}_{ij}(E)]$ in the following general manner: $\mathcal{I}_{ij}(E) \equiv \sum_{m=1}^{i+j} J_{2m+1} p^{2(n-m)} - \mathcal{I}_0 p^{2(i+j)}$, where $p = \sqrt{ME}$ and $\mathcal{I}_0 \equiv J_0 + i\frac{Mp}{4\pi}$ and the arbitrary parameters J_0 and J_{2m+1} $(m = 1, 2, \ldots)$ parametrize any sensible regularization/renormalization scheme. Generically, J_0 and J_{2m+1} should be independent of energy. Then $\mathcal{I}(E)$ takes the following form in ${}^1S_0$ channel ($L = 0$):

$$\mathcal{I}(E) \equiv -\mathcal{I}_0 U(p^2) U^T(p^2) + J_3 \Delta U_1(p^2) + J_5 \Delta U_2(p^2) + \cdots, \tag{6}$$

with

$$\Delta U_1(p^2) \equiv \frac{1}{p^2}\int_0^{p^2} \mathrm{d}t \frac{\mathrm{d}[U(t)U^T(t)]}{\mathrm{d}t}, \qquad \Delta U_{n+1}(p^2) \equiv \frac{1}{p^2}\int_0^{p^2} \mathrm{d}t \frac{\mathrm{d}[\Delta U_n(t)]}{\mathrm{d}t}, \quad n \geqslant 1. \tag{7}$$

[1] It is known that energy dependence in the potentials could be removed through unitary transformations, see e.g. [11].

While for $L \geqslant 1$, we have

$$\mathcal{I}(E) = (-\mathcal{I}_0 p^{2L} + J_3 p^{2L-2} + \cdots + J_{2L+1}) U(p^2) U^T(p^2) \\ + J_{2L+3} \Delta U_1(p^2) + J_{2L+5} \Delta U_2(p^2) + \cdots . \tag{8}$$

Obviously, any sensible prescription could be readily reproduced by assigning appropriate values to $J_{\ldots}$.

Some remarks are in order: first, all the divergent integrals involved assemble into the matrix $\mathcal{I}(E)$ of finite rank, or finitely many divergences are to be treated. This 'finiteness' should be able to substantiate the nonperturbative renormalization of T. Second, the parameters $[J_{\ldots}]$ in $\mathcal{I}(E)$ are nonperturbative and irreducible in the sense that they will appear as basic parameters in the compact form of T. Third, J_0 is very special as it always appears together with $\mathrm{i}\frac{Mp}{4\pi}$ in each entry of $\mathcal{I}(E)$ while $[J_{2m+1}, m > 0]$ do not.

The algebraic LSE could now be readily solved (the energy dependence in τ and $\mathcal{I}$ will be omitted below):

$$\tau = (1 - \lambda \circ \mathcal{I} \circ)^{-1} \lambda = \lambda (1 - \circ \mathcal{I} \circ \lambda)^{-1}. \tag{9}$$

Then the on-shell T for the channel L reads [9] (from now on we use $[C_{\ldots}]$ to denote $[\lambda_{\ldots}]$)

$$\frac{1}{\mathbf{T}_L} \equiv \left. \frac{1}{T_L(q, q')} \right|_{q=q'=p} = \mathcal{I}_0 + \frac{N_L([C_{\ldots}], [J_{2m+1}], p^2)}{D_L([C_{\ldots}], [J_{2m+1}], p^2) p^{2L}}, \tag{10}$$

where N_L and D_L are the polynomials in terms of real parameters: the contact couplings $[C_{\ldots}], [J_{2m+1}, m > 0]$ and p^2. While for the coupled channels $({}^3L_J - {}^3L'_J, L = J - 1, L' = J + 1)$, one could find the following [12]:

$$\mathbf{T}_J^{-1} = \mathcal{I}_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{\mathcal{N}_{L,L}}{\mathcal{D}_{L,L} p^{2L}}, & \frac{-\mathcal{N}_{L,L'}}{\mathcal{D}_{L,L'} p^{2J}} \\ \frac{-\mathcal{N}_{L,L'}}{\mathcal{D}_{L,L'} p^{2J}}, & \frac{\mathcal{N}_{L',L'}}{\mathcal{D}_{L',L'} p^{2L'}} \end{pmatrix}. \tag{11}$$

Again $[\mathcal{N}_{\ldots}, \mathcal{D}_{\ldots}]$ are the real polynomials in terms of $[C_{\ldots}], [J_{2m+1}, m > 0]$ and p^2. Note that such T-matrices are automatically unitary.

3. Renormalization of effective field theories in nonperturbative regimes

In the following, it suffices to mainly work with the uncoupled channels for unraveling the novel features of renormalization that elude the conventional perturbative scenario and wisdom.

3.1. On-shell cases

First, let us consider the on-shell case. The on-shell T-matrices given in equations (10) and (11) exhibit the following important features worth emphasis [9, 12, 13]: (1) first, the same complex parameter $\mathcal{I}_0$ appears in all channels in the same isolated position in $1/T$ or $\mathbf{T}^{-1}$, i.e. $\mathcal{I}_0$ is 'decoupled' from $[C_{\ldots}]$ and $[J_{2m+1}, m > 0]$ in every channel[2]. (2) Second, as is already noted above, only finitely many irreducible divergences $[J_{\ldots}]$ enter the game. That is, $\mathrm{Rank}(\mathcal{I}) < \infty$.

Since the p-dependence of the on-shell T-matrices is physical, the prescription variations (i.e. variations in $[J_{\ldots}]$) must be compensated for by that of the couplings. This is nothing else but the principle of RG invariance; then appropriate combinations of $[N_{\ldots}]$

[2] The rigorous proof of this point for the ${}^1S_0$ channel has been given in [9], which could be generalized to higher channels.

and $[D_{...}]$ must be RG invariants. Moreover, the isolation of $\mathcal{I}_0$ in all T^{-1}'s makes it alone an RG invariant parameter to be physically determined [9, 12, 13]. Therefore, *in the nonperturbative regime*, J_0 becomes a universal physical scale in the low energy NN scattering. This is not a bizarre event: in the Wilsonian approach, the nontrivial fixed-point solution [14] just equals the negative inverse of J_0 computed in the cut-off scheme: $(\hat{V}_0(p))^{-1} = \frac{M}{2\pi^2}\left(-\Lambda + \frac{p}{2}\ln\frac{\Lambda+p}{\Lambda-p}\right) = -J_0(p,\Lambda) = -\mathrm{Re}(\mathcal{I}_{0;\Lambda})$. There is only one exception at leading order in ${}^1S_0$ where J_0 mixes with $1/C_0$ [5].

There are also some cases at lower orders where some divergences in $[J_{2m+1}, m \neq 0]$ might be absorbed into the couplings. For example, at $\Delta = 2$, the inverse on-shell T for ${}^1S_0$ [10] reads

$$\frac{1}{T} = \mathcal{I}_0 + \frac{N_0}{D_{0;0} + D_{0;1}p^2}, \tag{12}$$

with

$$N_0 = (1 - C_2J_3)^2, \qquad D_{0;0} = C_0 + C_2^2J_5, \qquad D_{0;1} = C_2(2 - C_2J_3). \tag{13}$$

It could be made finite by requiring $\frac{D_{0;0}}{N_0}$ and $\frac{D_{0;1}}{N_0}$ to be finite constants: $\frac{1}{T} = \mathcal{I}_0 + \frac{1}{c_0+2c_2p^2}$. The solutions are quite sophisticated [9],

$$\begin{aligned} C_2^{(\pm)} &= J_3^{-1}(1 \pm (1 + 2c_2J_3)^{-1/2}), \\ C_0^{(\pm)} &= \frac{c_0}{1 + 2c_2J_3} - \frac{J_5}{J_3^2}(1 \pm (1 + 2c_2J_3)^{-1/2})^2. \end{aligned} \tag{14}$$

However, there is no way to subtract the divergence in J_0 with such sophisticated counterterms; thus, the counterterm algorithm fails here.

Things get worse at higher orders. For example, at $\Delta = 4$ for ${}^1S_0$, we have [9]

$$\frac{1}{T} = \mathcal{I}_0 + \frac{N_0 + N_1p^2 + N_2p^4}{D_0 + D_1p^2 + D_2p^4 + D_3p^6} \tag{15}$$

with $N_2 = C_4^2J_3^2$, $D_3 = -C_4^2J_3$, where it is simply impossible to renormalize N_2 and D_3 with counterterms from couplings at the same time as $N_2/D_3 = -J_3$ contains a divergence! This status is generically true at higher orders, regardless of channels. Then, in order to obtain finite results, we have to go beyond the counterterms from couplings. All these points hint to us about something unprecedented. To unravel them, we turn to the off-shell case.

3.2. Off-shell cases

As already pointed out above, it suffices to consider the renormalization of τ. To expose the most crucial nonperturbative structures, let us turn equation (9) into the following inverse form:

$$\tau^{-1} = \lambda^{-1} - \mathcal{I}, \tag{16}$$

in terms of which the unitarity $\tau^{-1} - (\tau^\dagger)^{-1} = \frac{\mathrm{i}Mp}{2\pi}U(p)U^T(p)$ is obviously not affected by renormalization at all. Since the p-dependence of T is physical, so is it for τ. Thus τ must be finite and prescription-independent. Then equation (16) tells us that the renormalization of τ ultimately reduces to the removal of divergences in $\mathcal{I}$ in such a manner that the combination $\lambda^{-1} - \mathcal{I}$ is RG invariant or physical.

At first sight, this seems trivial as one could let λ^{-1} absorb all the divergences, i.e. counterterms from λ^{-1} are at work. Unfortunately, this is not true due to the following two intrinsic mismatches between λ^{-1} and $\mathcal{I}$: (i) λ^{-1} is constrained as follows due to the truncation constraint (3):

$$(\lambda^{-1})_{ij} = 0, \qquad \forall\, i, j : i + j \leqslant \Delta/2 - L, \tag{17}$$

while $\mathcal{I}$ is free from such constraints; (ii) $\mathcal{I}$ is energy dependent while λ^{-1} is not.

Let us elaborate. According to the EFT power counting, the counterterms must also be constrained by equation (17). Then there is no counterterm for the entries $\mathcal{I}_{ij}$ where $(\lambda^{-1})_{ij} = 0$ to subtract the divergences there, that is, the counterterm algorithm could not perform sufficient subtractions. For example, for ${}^1S_0$ at $\Delta = 2$, we have

$$\lambda = \begin{pmatrix} C_0 & C_2 \\ C_2 & 0 \end{pmatrix} \quad \Rightarrow \quad \lambda^{-1} = \begin{pmatrix} 0 & C_2^{-1} \\ C_2^{-1} & -C_0 C_2^{-2} \end{pmatrix}. \tag{18}$$

In the meantime,

$$\mathcal{I} = \begin{pmatrix} -\mathcal{I}_0 & J_3 - \mathcal{I}_0 p^2 \\ J_3 - \mathcal{I}_0 p^2 & J_5 + J_3 p^2 - \mathcal{I}_0 p^4 \end{pmatrix}. \tag{19}$$

It is obvious that the divergence in $\mathcal{I}_{0,0}$ (i.e. in J_0) could in no way be subtracted by counterterms from $(\lambda^{-1})_{0,0}$, which is zero as required by the consistent EFT power counting. There is no sensible way within nonperturbative regimes to remove such inherent mismatches between the EFT power counting and the divergence 'configuration'; hence, the counterterms from EFT couplings could not work here. This can be seen as follows: suppose we introduce higher order terms in the potential so that the mismatch is gone, for the example considered above, it means $\lambda_{1,1} \neq 0$ or the term $\sim q^2 q'^2$ is included and hence $(\lambda^{-1})_{0,0} \neq 0$, then the general principle of EFT power counting is broken, according to which terms $\sim q^4$ and $\sim q'^4$ should also be included, which means $\lambda_{2,0} = \lambda_{0,2} \neq 0$. It will not help by including $\lambda_{2,0}$ and $\lambda_{0,2}$ as then λ and hence $\mathcal{I}$ will be enlarged, and the mismatch will persist between the enlarged λ^{-1} and $\mathcal{I}$, unless one 'removes' the truncation itself, which is actually impossible in any EFT approach.

It will also not help even if one ignores the EFT power counting in constructing counterterms, since equation (16) means the counterterms would necessarily develop energy dependence as the divergences in $\mathcal{I}$ are energy dependent, which is theoretically unfavorable. This is due to the fact that the 'nonperturbative' counterterms introduced through λ^{-1} would lead to non-polynomial energy dependence in the local couplings, that is, the operators that are 'nonlocal' in time, not the 'local' ones that are allowed within a contact potential approach. In other words, even if one works with an energy-dependent version of the potentials (or operators 'local' in time), there still may appear mismatches between the inverse λ^{-1}, which is now non-polynomial in terms of energy, and the integrals $\mathcal{I}$, which are polynomials in terms of energy. Therefore, such choices could not remove the mismatches unraveled; actually, it leads to new serious problems. As the mismatches only originated from the nonperturbative structures of the divergences involved, we suspect that they may also persist in the more realistic cases with nonlocal potentials, especially for the cases with pion exchanges that interest most practitioners in the EFT approach to nuclear forces. Of course, definite conclusions are not available before rigorous solutions of such cases are available.

At this stage, we note that there is an exception at leading order ($\Delta = 0$) where $\lambda = C_0, \mathcal{I} = -\mathcal{I}_0$ and the mismatches between λ^{-1} and $\mathcal{I}$ are gone. Since the divergence status is not altered after one-pion exchange is included, this could explain why the counterterm algorithm works in such cases [2, 15, 16].

The intrinsic relations between EFT power counting and nonperturbative structures of divergences naturally led us to conclude that, beyond leading order, it is generally impossible to implement the counterterms from couplings in nonperturbative regimes. However, this is not equivalent to the failure of renormalization itself. In this connection, we recall that the ultimate goal of renormalization is to obtain finite amplitudes generated with EFT propagators and vertices, not how the divergences are removed, and that the most crucial step is to fix the undetermined constants generated in any renormalization procedure by imposing appropriate boundary conditions. Due to the difficulties described above, one is naturally led to the subtraction directly performed on the integrals in $\mathcal{I}$, or through other means that could yield equivalent effects. By whatever means, the final outcome is that, due to the energy or p-dependence of $\mathcal{I}$ (cf equation (6) or (8)), at least the constant J_0 (which is energy- or p-independent) could no longer mix with the couplings in λ^{-1} and therefore must be physically determined through imposition of appropriate boundary conditions. Thus the two mismatches between λ^{-1} and $\mathcal{I}$ lead to the RG invariance of J_0. In fact, as long as a $J_{2m+1}(m > 0)$ appears as a coefficient of a p-dependent matrix $U(p^2)U^T(p^2)$ or $\Delta U_n(p^2)(n \geqslant 1)$, it must also be physically determined. Thus we reproduced the same conclusion as obtained in the on-shell case.

4. Discussions and summary

Thus through the above analysis within the realm of contact potentials or EFT($\not\pi$), we showed that the conventional counterterm algorithm and the associated wisdom could not work beyond the leading order due to the intrinsic relations between EFT power counting and nonperturbative divergences. Then one must resort to other approaches beyond the counterterm algorithm. Here, we note that the counterterm algorithm refers to the construction of counterterms from EFT vertices or potentials, not the subtraction in general sense. Thus, in the treatments of NN scattering at higher orders where rigorous and explicit parametrization of the divergences is impossible and counterterms could not work, keeping the cut-off finite (which is one kind of subtraction already) and properly fine-tuning it together with other contact couplings is a choice that is pragmatic and reasonable [17, 18]. Or, one may choose some 'perturbative' treatment of the potentials beyond leading order as long as the convergence is assured [19]. For further progress, the nonperturbative properties and mismatches revealed here should be illuminating and hence carefully taken into account.

Before closing our presentation, let us expose another interesting point associated with the inverse formalism that is intrinsically nonperturbative: the EFT power counting expressed in terms of λ^{-1} seems somewhat 'unusual' due to the constraint (17). There are some entries that jump to zero and deviate from the seemingly well-ordered sequence of nonzero entries. This is again due to the truncation of potential which is natural from the EFT side. Further exploration of this point will be pursued in the future.

In summary, we provided a somewhat transparent analysis of the renormalization of EFT in nonperturbative regimes within the context of contact potentials or pionless EFT without introducing any deformation of the standard field-theoretical framework. In a formulation that makes the main structural issues lucid and obvious, it was shown for the first time that the intrinsic mismatches between EFT power counting and nonperturbative divergences preclude counterterm algorithms from being at work. Possible ways out and the reasonable aspects of some approaches were also briefly addressed. The notions revealed here could well be applied to wider range of physical systems that are dominated by singular short-distance interactions.

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